NONCOMMUTATIVE BROWNIAN MOTIONS ASSOCIATED WITH KESTEN DISTRIBUTIONS AND RELATED POISSON PROCESSES

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Abstract. We introduce and study a noncommutative two-parameter family of noncommutative Brownian motions in the free Fock space. They are associated with Kesten laws and give a continuous interpolation between Brownian motions in free probability and monotone probability. The combinatorics of our model is based on ordered non-crossing partitions, in which to each such partition P we assign the weight $w(P) = p^{e(P)}q^{e'(P)}$, where $e(P)$ and $e'(P)$ are, respectively, the numbers of *disorders* and *orders* in P related to the natural partial order on the set of blocks of P implemented by the relation of being inner or outer. In particular, we obtain a simple relation between Delaney's numbers (related to inner blocks in non-crossing partitions) and generalized Euler's numbers (related to orders and disorders in ordered non-crossing partitions). An important feature of our interpolation is that the mixed moments of the corresponding creation and annihilation processes also reproduce their monotone and free counterparts, which does not take place in other interpolations. The same combinatorics is used to construct an interpolation between free and monotone Poisson processes.

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1. Introduction

In noncommutative probability several noncommutative Brownian motions have been introduced and studied. In particular, different notions of noncommutative independence (either abstract, or restricted to the Fock space level) lead to different noncommutative (i) central limit theorems, (ii) analogues of the classical Brownian motion, (iii) Poisson-type processes. In this context let us mention here the well known examples of the boson Brownian motion on the symmetric Fock space of Hudson and Parthasarathy [10], the fermion Brownian motion on the antisymmetric Fock space of Applebaum and Hudson [1], the free Brownian motion on the free Fock space of Speicher [22] and the monotone Brownian motion of Muraki [18] (see also Lu [17]) on the monotone Fock space. An important feature of the free Brownian motion is that it can be obtained as the limit in distribution of (as the dimension becomes infinite) of a sequence of Brownian motions in the finite-dimensional Hermitian matrices, as shown by Biane [4].

Interpolations between these examples have also been studied. For instance, an interpolation between the boson, fermion and free Brownian motions called the q-Brownian motion was studied by Bożejko and Speicher [6]. In the bialgebra and Hopf algebra context, respectively, two different q-central limit theorems and related Brownian motions

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were studied by Schurmann [29] and the author [14,15]. In this paper, we introduce and study an interpolation between noncommutative Brownian motions in monotone probability of Muraki [19] and free probability of Voiculescu [22,23].

Our interpolation depends on two continuous nonnegative parameters p, q. The most important reason why we find this interpolation interesting is that it is based on the combinatorics of ordered non-crossing partitions $\mathcal{ONC}$ and thus gives a very natural interpolation between the combinatorics of non-crossing partitions $\mathcal{NC}$ in free probability [21] and the combinatorics of monotone non-crossing partitions $\mathcal{MNC}$ in monotone probability [19].

By a monotone non-crossing partition of the set $[n] := \{1, 2, \ldots, n\}$ we understand a sequence $P = (P_1, P_2, \ldots, P_r)$ of blocks, in which the fact that block P_j is inner with respect to block P_i implies that $i < j$. In other words, the natural partial order on the set of blocks of P implemented by the relation of being inner or outer is respected by the formal order defined by positions of blocks in P. The term 'monotone non-crossing partition' comes from our work [16], but it can be traced back to the partitions studied by Muraki [19].

The key observation is that the class of non-crossing partitions as well as the class of monotone non-crossing partitions can be included in the scheme of ordered non-crossing partitions if one assigns the weight $p^{e(P)}$ to each $P \in \mathcal{ONC}$, where $p \geqslant 0$ and $e(P)$ is the number of *disorders*, or *Euler inversions*. By a disorder in $P = (P_1, P_2, \ldots, P_r)$ we understand a pair of blocks $\{P_i, P_j\}$ such that $i < j$, P_i is inner with respect to P_j and lies immediately under P_j, i.e. with no intermediate blocks between. Then, for $p = 0$ we obtain monotone non-crossing partitions with the same weight and the case $p = 1$ corresponds to all ordered non-crossing partitions with the same weight, which reduces to $\mathcal{NC}$.

In a similar way, we can introduce the second parameter $q \geqslant 0$ and assign to each P the weight $q^{e'(P)}$, where $e'(P)$ is equal to the number of *orders* in P (by an order we understand any pair $\{P_i, P_j\}$ such that $i < j$ and P_i is outer with respect to P_j). The second parameter is not necessary to give all Kesten laws as such (in particular, the Wigner law) but is needed when to reproduce all mixed moments of the free Brownian motion. Besides, it provides a natural symmetry between orders and disorders in our combinatorics. As a consequence, we obtain a nice relation between *Delaney's numbers* $\mathcal{D}(n, k+j)$, which give the numbers of partitions $\pi \in \mathcal{NC}_{2n}^2$ which have $k + j$ inner blocks, and *generalized Euler's numbers* $\mathcal{E}(n, k, j)$, by which we understand the numbers of ordered partitions $P \in \mathcal{ONC}_{2n}^2$ which have k disorders and j orders.

Using a weight function on $L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$ related to $w(P)$, we define on the free Fock space (p, q)-creation and annihilation processes, $(a_t)_{t \geqslant 0}$ and $(a_t^*)_{t \geqslant 0}$, respectively, and the corresponding canonical position process $(\omega_t)_{t \geqslant 0}$, or Brownian motion, where $\omega_t = a_t + a_t^*$ (parameters p, q are supressed in the notation) (in a similar way we can define the canonical momentum process $\eta_t = i(a_t - a_t^*)$). In particular, we obtain the combinatorial formula

$$\varphi(\omega_t^{2n}) = \sum_{P \in \mathcal{ONC}_{2n}^2} \frac{w(P) t^n}{b(P)!} \tag{1.1}$$

for the even moments of the position process in the vacuum state φ on $\mathcal{F}(\mathbb{R}_+)$ (the odd moments vanish), where the weight is given by

$$w(P) = p^{e(P)}q^{e'(P)} \tag{1.2}$$

and $b(P)$ denotes the number of blocks of P. By $\mathcal{ONC}_n$ ($\mathcal{ONC}_n^2$) we denote the set of ordered non-crossing partitions (pair-partitions) of $[n]$.

Similarly, the moments of suitably defined processes of Poisson type $(\gamma_t)_{t\geqslant 0}$ (dependence on p,q is supressed again) can be expressed as

$$\varphi(\gamma_t^n) = \sum_{P\in\mathcal{ONC}_n} \frac{w(P)t^{b(P)}}{b(P)!}. \tag{1.3}$$

It is easy to see that for $(p,q) = (0,1)$ and $(p,q) = (1,1)$ the above formulas give the moments of canonical position processes and Poisson processes in monotone probability and free probability, respectively. The case $(p,q) = (1,0)$ corresponds to the anti-monotone processes. Finally, the case $(p,q) = (0,0)$ corresponds to boolean processes.

Moreover, we show that our canonical position processes are associated with Kesten distributions. For instance, for $t = 1$, the moments of ω_1 agree with the moments of Kesten measures with densities

$$f_{p,q}(x) = \frac{1}{\pi}\frac{\sqrt{2(p+q)-x^2}}{2-(2-p-q)x^2}, \quad x \in [-\sqrt{2(p+q)}, \sqrt{2(p+q)}] \tag{1.4}$$

and atoms at $x = \pm\frac{1}{\sqrt{1-(p+q)/2}}$ for $p+q < 1$. In particular, for $(p,q) = (0,1)$ (as well as for $(p,q) = (1,0)$) we obtain the standard arcsine law and for $(p,q) = (1,1)$ – the standard Wigner law.

Let us point out that our model interpolates not only between the moments of the canonical free and monotone position processes, but also between the corresponding mixed moments of creation and annihilation processes. Moreover, it reproduces independence on both Fock spaces. Therefore, at least on the Fock space level, it may be viewed as an interpolation between monotone independence and free independence. This feature is absent in the t-interpolation of Bożejko and Wysoczański [7], which also reproduces the moments of Kesten laws [11], but does not reproduce monotone independence. Namely, neither the mixed moments of monotone independent creation and annihilation processes nor the mixed moments of position processes (with arcsine distributions) associated with disjoint intervals agree with the corresponding moments of t-deformed processes for the right value of t.

To put it in the general framework of interacting Fock spaces, let us notice that one of the main points of the t-interpolation and of the 'gaussianization of probability measures' [2] is that one can reproduce the moments of classical probability measures by means of noncommutative Gaussian operators on the *one-mode interacting Fock spaces* (see also [8,9] for a related result on symmetric measures). Of course, one-mode interacting Fock spaces are examples of interacting Fock spaces [3] in which deformations of the inner product on the free Fock space are very simple (and are related to the Jacobi parameters of probability measures). Our deformations of the free Fock spaces (or, of the corresponding creation and annihilation operators) are more complicated and, roughly speaking, they might be viewed as examples of *two-mode*

interacting Fock spaces. Although our motivation is of combinatorial nature rather than related to the interacting Fock space structure, it seems that this is the reason why we can also reproduce noncommutative independence apart from the classical properties like 'gaussianization' in the one-mode case.

Let us mention here that a discrete interpolation between monotone probability and free probability, called the *monotone hierarchy of freeness*, was studied in [16]. In particular, we obtained a combinatorial formula for the mixed moments of the hierarchy of m-monotone Gaussians, based on the combinatorics in which one counts blocks which are inner with respect to each block (as in the case of Poisson operators studied by Muraki [19]) rather than on the combinatorics based on blocks' depths [2], or levels of Catalan paths [8,9], which is equivalent in the case of symmetric measures. Moreover, this also allowed us to reproduce the mixed moments of monotone independent creation and annihilation operators, which are not reproduced by the formulas given in [2].

2. Combinatorics of Kesten laws

The set of all partitions of the set $[n]$ will be denoted $\mathcal{P}_n$. We say that $\pi \in \mathcal{P}_n$ is *non-crossing* if there are no pairs $\{k, k'\} \subset \pi_i$ and $\{m, m'\} \subset \pi_j$ with $i \neq j$ and such that $k < m < k' < m'$. The set of all non-crossing partitions of the set $[n]$ will be denoted $\mathcal{NC}_n$. By $\mathcal{P}_n^2$ we will denote the set of all *pair-partitions* of $[n]$ and $\mathcal{NC}_n^2 := \mathcal{NC}_n \cap \mathcal{P}_n^2$.

On the set of blocks of $\pi \in \mathcal{NC}_n$ we can introduce a natural partial order. Namely, we will say that π_i is *inner* with respect to π_j for $i \neq j$ if there exist $a, b \in \pi_j$ such that for all $c \in \pi_i$ it holds that $a < c < b$, in which case we shall write $\pi_j < \pi_i$. Moreover, we set $\pi_j \leqslant \pi_i$ iff $\pi_j < \pi_i$ or $\pi_j = \pi_i$. Equivalently, we will say that π_j is *outer* w.r.t. π_i. We will say that blocks π_j, π_i are *neighboring* if they are comparable in the above sense and there are no other 'intermediate' blocks between them, i.e. if $\pi_j < \pi_i$ and $\pi_j \leqslant \pi_k \leqslant \pi_i$ implies that $k = i$ or $k = j$. If π_i, π_j are neighboring blocks and $\pi_j < \pi_i$, we will write $\pi_j \prec \pi_i$. A block π_i is called *outer* in π if π has no blocks which are outer with respect to π_i.

The pair $P = (\pi, \sigma)$, where $\pi = \{\pi_1, \pi_2, \ldots, \pi_k\} \in \mathcal{P}_n$ and σ is a permutation from the symmetric group S_k, will be called an *ordered partition* of the set $[n]$ and will be identified with the sequence $P = (P_1, P_2, \ldots, P_k)$, where $P_i = \pi_{\sigma(i)}$. In particular, we will write $P_j \prec P_i$ if $\pi_{\sigma(j)} \prec \pi_{\sigma(i)}$. The set of all ordered (pair, non-crossing, non-crossing pair) partitions of $[n]$ will be denoted $\mathcal{O}P_n$ (respectively, $\mathcal{O}P_n^2$, $\mathcal{ONC}_n$, $\mathcal{ONC}_n^2$).

Let us observe that in each ordered partition $P = (\pi, \sigma)$, the permutation σ defines a linear order on the set of blocks of π. Comparing this order with the partial order given by the relation of being inner (outer) for $P \in \mathcal{ONC}_n$, we can introduce „disorders" betwen blocks.

Definition 2.1. If $P = (P_1, P_2, \ldots, P_k) \in \mathcal{ONC}_n$, we will say that the pair $\{P_i, P_j\}$ forms a *disorder* or *Euler inversion* if $i < j$ and $P_j \prec P_i$. If $i < j$ and $P_i \prec P_j$, we will say that the pair $\{P_i, P_j\}$ forms an *order*. The numbers of all disorders and orders in P will be denoted $e(P)$ and $e'(P)$, respectively. In a similar way we define disorders and orders in the permutation $\sigma \in S_n$ associated with each index $i \in \{1, 2, \ldots, n-1\}$ for which $\sigma(i) < \sigma(i+1)$ and $\sigma(i) > \sigma(i+1)$, respectively. The numbers of all disorders and orders in σ will be denoted $e(\sigma)$ and $e'(P)$, respectively.

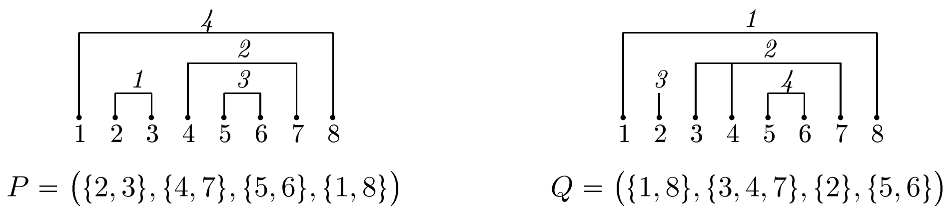


FIGURE 1. Examples of ordered non-crossing partitions

Remark 2.1. Well-known Euler numbers, denoted $\left\langle {n \atop k} \right\rangle$, give the numbers of permutations of the set $[n]$ which have k disorders.

Example 2.1. Consider ordered non-crossing partitions $P \in \mathcal{ONC}_8^2$ and $Q \in \mathcal{ONC}_8$ given in Fig.1. Parititon P has 3 pairs of neighboring blocks: $P_4 \prec P_2$, $P_4 \prec P_1$ and $P_2 \prec P_3$. Only the first two give disorders, thus $e(P) = 2$ and $e'(P) = 1$. Pairs of neighboring blocks in Q are the following: $Q_1 \prec Q_2$, $Q_1 \prec Q_3$ and $Q_2 \prec Q_4$. None of them gives a disorder, thus $e(Q) = 0$ and $e'(Q) = 3$. Non-crossing ordered paritions which do not have disorders are called *monotone* [16].

By $\mathcal{ONCC}_n$ ($\mathcal{ONCC}_n^2$) we denote the set of those ordered non-crossing partitions (pair-partitions) of $[n]$, in which the numbers 1 and n belong to the same block. Such partitions will be called *covered*. Let us introduce numbers

$$r_n = \frac{1}{n!} \sum_{P \in \mathcal{ONC}_{2n}^2} w(P) \ , \qquad s_n = \frac{1}{n!} \sum_{P \in \mathcal{ONCC}_{2n}^2} w(P)$$

for $n \geqslant 1$ and set $r_0 = 1$, $s_0 = 0$, where $w(P)$ is given by (1.2).

Proposition 2.1. *The following relation between sequences $(r_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ holds:*

$$r_n = \sum_{m=1}^{n} \sum_{k_1+k_2+\ldots+k_m=n} s_{k_1} s_{k_2} \ \ldots \ s_{k_m} \ , \qquad n \geqslant 1.$$

where the second summation runs over positive indices $k_1, \ldots, k_m$.

Proof. Let $Q^{(i)} = (Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_{k_i}^{(i)})$, where $i = 1, 2, \ldots, m$, be arbitrary pair-partitions from the sets $\mathcal{ONCC}_{2k_i}^2$, respectively, such that $k_1 + \ldots + k_m = n$. From the blocks of all these partitions we construct ordered pair partitions of $[n]$ with the shape given by Fig.2. We order all blocks $Q_j^{(i)}$ of the subpartition $Q^{(i)}$ in such a way that the order between blocks from the same partition $Q^{(i)}$ is preserved. There are $\frac{n!}{k_1! \ldots k_m!}$ such orderings and each of them defines exactly one $P \in \mathcal{ONC}_{2n}^2$. Moreover, each partition from $\mathcal{ONC}_{2n}^2$ can be obtained in this fashion by an appropriate choice of covered partitions $Q^{(i)}$. From the above reasoning we obtain

$$|\mathcal{ONC}_{2n}^2| = \sum_{m=1}^{n} \sum_{k_1+k_2+\ldots+k_m=n} \frac{n!}{k_1! \ldots k_m!} |\mathcal{ONCC}_{2k_1}^2| \ldots |\mathcal{ONCC}_{2k_m}^2| \,. \tag{2.1}$$

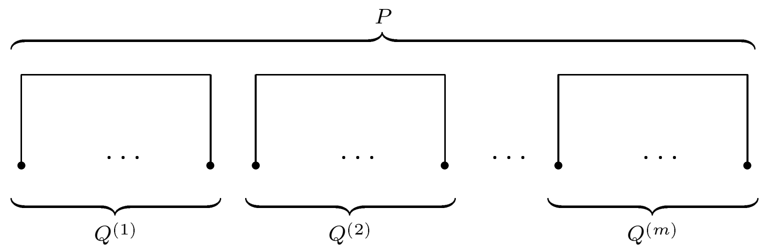


FIGURE 2. $P \in \mathcal{ONC}$ constructed from $Q^{(1)}, \ldots, Q^{(m)} \in \mathcal{ONCC}$.

Clearly, between blocks which belong to different partitions $Q^{(i)}$ and $Q^{(j)}$, $i \neq j$, there are no orders or disorders. Therefore,

$$\begin{aligned} e(P) &= e(Q^{(1)}) + e(Q^{(2)}) + \ldots + e(Q^{(r)}) \\ e'(P) &= e'(Q^{(1)}) + e'(Q^{(2)}) + \ldots + e'(Q^{(r)}) \end{aligned}$$

which gives multiplicativity of the weights

$$w(P) = w(Q^{(1)})w(Q^{(2)}) \ldots w(Q^{(r)})$$

and that, together with (2.1), gives the assertion. ■

Corollary 2.1. *Let $R(z) = \sum_{n=0}^{\infty} r_n z^n$ and $S(z) = \sum_{n=0}^{\infty} s_n z^n$ be formal power series. Then*

$$R(z) = \frac{1}{1 - S(z)}\,.$$

Proof. Using Proposition 2.1 and simple algebraic computations, we get

$$\begin{aligned} R(z) - 1 &= \sum_{n=1}^{\infty} r_n z^n = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{k_1+k_2+\ldots+k_m=n} s_{k_1} s_{k_2} \ldots s_{k_m} z^n \\ &= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \sum_{k_1+k_2+\ldots+k_m=n} s_{k_1} s_{k_2} \ldots s_{k_m} z^n \\ &= \sum_{m=1}^{\infty} (S(z))^m = \frac{S(z)}{1 - S(z)} \end{aligned}$$

from which our assertion follows. ■

In order to find $R(z)$, we introduce another sequence of numbers, denoted $(a_n)_{n=0}^{\infty}$ and defined by the combinatorial formula

$$a_n = \sum_{m=1}^{n} \sum_{k_1+k_2+\ldots+k_m=n} p^m s_{k_1} s_{k_2} \ldots s_{k_m} \tag{2.2}$$

for $n \geqslant 1$ and we set $a_0 = 1$. Let us observe that a_n is the sum of contributions to r_{n+1} of these pair partitions $P = (P_1, \ldots, P_{n+1}) \in \mathcal{ONCC}^2_{2n+2}$ which are covered by the block

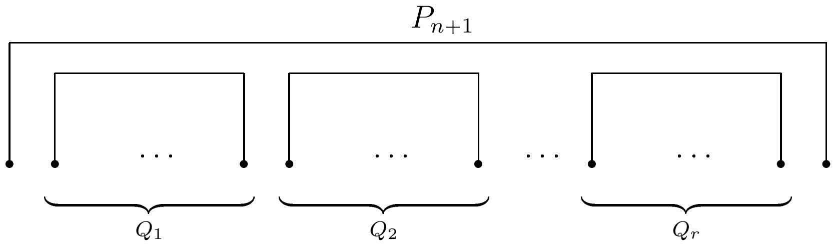


FIGURE 3. Partition covered by the last block P_{n+1}.

of highest color, namely P_{n+1}. Therefore,

$$a_n = \frac{1}{n!} \sum_{\substack{P \in \mathcal{ONCC}^2_{2n+2} \\ P_{n+1}=\{1,2n+2\}}} w(P) \tag{2.3}$$

for $n \geqslant 0$. In fact, each neighboring block of P_{n+1} corresponds to a certain covered partition $Q_i \in \mathcal{ONCC}^2_{2k_i}$ for $i = 1, \ldots, m$, as Fig.3 shows. Of course, $k_1 + \ldots + k_m = n$. Besides, let us observe that each neighboring block of P_{n+1} forms a disorder with block P_{n+1}. This justifies equivalence of (2.2) and (2.3).

Corollary 2.2. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be a formal power series. Then*

$$A(z) = \frac{1}{1 - pS(z)}.$$

Proof. Using algebraic calculations and equation (2.2), we obtain

$$\begin{aligned} A(z) - 1 &= \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{k_1+k_2+\ldots+k_m=n} p^m s_{k_1} s_{k_2} \ldots s_{k_m} z^n \\ &= \sum_{m=1}^{\infty} p^m \sum_{n=m}^{\infty} \sum_{k_1+k_2+\ldots+k_m=n} s_{k_1} s_{k_2} \ldots s_{k_m} z^n \\ &= \sum_{m=1}^{\infty} (pS(z))^m = \frac{pS(z)}{1-pS(z)} \end{aligned}$$

from which we get the assertion. ■

To find a relation between $R(z)$ and $A(z)$ we shall need some additional notations. Let $\mathcal{ONCC}_n(r)$ ($\mathcal{ONCC}^2_n(r)$) denote the sets of ordered partitions (pair partitions) of $[n]$ with r outer blocks, and, for $r = 1, \ldots, n$, introduce sequences

$$s_n^{(r)} = \frac{1}{n!} \sum_{Q \in \mathcal{ONCC}^2_{2n}(r)} w(P) \tag{2.4}$$

for $n \geqslant 1$, and set $s_0^{(r)} = 0$. Of course, $s_n^{(1)} = s_n$ and $s_r^{(r)} = 1$.

Proposition 2.2. *Let $S^{(r)}(z) = \sum_{n=r}^{\infty} s_n^{(r)} z^n$, where $r \geqslant 1$. Then*

$$S^{(r)}(z) = \big(S(z)\big)^r$$

Proof. Notice that we have the following relation between sequences $(s_n)_{n\geqslant 1}$ and $(s_n^{(r)})_{n\geqslant 1}$:

$$s_n^{(r)} = \sum_{k_1+k_2+\ldots+k_r=n} s_{k_1}\ldots s_{k_r}\,, \qquad n\geqslant r\,,$$

where $k_1,k_2,\ldots,k_r$ are assumed to be positive integers. This gives

$$S^{(r)}(z)=\sum_{n=r}^{\infty}\sum_{k_1+k_2+\ldots+k_r=n} s_{k_1}\ldots s_{k_r}z^n = \Big(\sum_{n=1}^{\infty}s_n z^n\Big)^r = \big(S(z)\big)^r\,.$$

which proves our assertion. ■

Lemma 2.1. *For $n\geqslant r+1$ it holds that*

$$s_n^{(r)} = \frac{r}{n}\sum_{k=1}^{n-r+1} a_{k-1}s_{n-k}^{(r-1)} \;+\; \frac{q}{n}\sum_{k=1}^{n-r}(2n-2k-r)a_{k-1}s_{n-k}^{(r)}\,.$$

Proof. Let us split (2.4) into two sums: the first one running over those partitions from $\mathcal{ONCC}_{2n}^2(r)$ in which the block of highest color is outer, and the second one – over the remaining partitions. Then

$$\begin{aligned}
s_n^{(r)} &= \frac{1}{n!}\sum_{\substack{P\in\mathcal{ONCC}_{2n}^2(r)\\ P_n-outer}} w(P) \;+\; \frac{1}{n!}\sum_{\substack{P\in\mathcal{ONCC}_{2n}^2(r)\\ P_n-inner}} w(P)\\
&= \frac{1}{n!}\sum_{k=1}^{n-r+1}\sum_{\substack{T\in\mathcal{ONCC}_{2k}^2\\ T_k=\{1,2k\}}}\sum_{Q\in\mathcal{ONCC}_{2n-2k}^2(r-1)} r\binom{n-1}{k-1}w(T)w(Q)\\
&+ \frac{1}{n!}\sum_{k=1}^{n-r}\sum_{\substack{T\in\mathcal{ONCC}_{2k}^2\\ T_k=\{1,2k\}}}\sum_{Q\in\mathcal{ONCC}_{2n-2k}^2(r)} (2n-2k-r)\binom{n-1}{k-1}qw(T)w(Q)\\
&= \frac{r}{n}\sum_{k=1}^{n-r+1} a_{k-1}s_{n-k}^{(r-1)} \;+\; \frac{q}{n}\sum_{k=1}^{n-r}(2n-2k-r)a_{k-1}s_{n-k}^{(r)}\,.
\end{aligned}$$

which completes the proof. ■

From the above lemma we get a relation between the $S^{(r)}(z)$'s and $A(z)$ in the form of a differential equation.

Corollary 2.3. *The functions $S^{(r)}(z)$ and $A(z)$ satisfy the differential recurrence*

$$\big(S^{(r)}(z)\big)' = rS^{(r-1)}(z)A(z) + 2qz\big(S^{(r)}(z)\big)'A(z) - qrA(z)S^{(r)}(z)\,,$$

with initial conditions $S^{(r)}(0)=0\,, A(0)=1$.

Proof. In view of Lemma 2.1, we have

$$\begin{aligned}
\big(S^{(r)}(z)\big)' &= \sum_{n=r}^{\infty} ns_n^{(r)}z^{n-1}\\
&= rz^{r-1} + r\sum_{n=r+1}^{\infty}\sum_{k=1}^{n-r+1} a_{k-1}s_{n-k}^{(r-1)}z^{n-1}
\end{aligned}$$

$$
\begin{aligned}
& +q\sum_{n=r+1}^{\infty}\sum_{k=1}^{n-r}(2n-2k-r)a_{k-1}s_{n-k}^{(r)}z^{n-1} \\
= \; & r\sum_{n=r}^{\infty}\sum_{k=1}^{n-r+1}a_{k-1}s_{n-k}^{(r-1)}z^{n-1}+2q\sum_{n=r+1}^{\infty}\sum_{k=1}^{n-r}(n-k)a_{k-1}s_{n-k}^{(r)}z^{n-1} \\
& -qr\sum_{n=r+1}^{\infty}\sum_{k=1}^{n-r}a_{k-1}s_{n-k}^{(r)}z^{n-1} \\
= \; & r\sum_{n=r-1}^{\infty}\sum_{k=0}^{n-(r-1)}a_{k}s_{n-k}^{(r-1)}z^{n}+2qz\sum_{n=r}^{\infty}\sum_{k=0}^{n-r}a_{k}(n-k)s_{n-k}^{(r)}z^{n-1} \\
& -qr\sum_{n=r}^{\infty}\sum_{k=0}^{n-r}a_{k}s_{n-k}^{(r)}z^{n} \\
= \; & rS^{(r-1)}(z)A(z)+2qz\big(S^{(r)}(z)\big)'A(z)-qrA(z)S^{(r)}(z)
\end{aligned}
$$

Of course, $S^{(r)}(0)=0$ and $A(0)=a_0=1$, which completes the proof. ■

Using the relations bewteen functions $R(z), A(z)$ and $S^{(r)}(z)$, $r\geqslant 1$, we can now derive the explicit form of $R(z)$, which will turn out to be related to the moment generating function of Kesten laws.

Theorem 2.1. *For each $p\geqslant 0, p\geqslant 0$, $p+q>0$, the sequence $(m_n)_{n\geqslant 0}$, where*

$$
m_n=\begin{cases} r_k & if\;\; n=2k \\ 0 & if\;\; n\;\; is\;\; odd\end{cases}
$$

is the sequence of moments of the Kesten measure (1.4). The probability measure determined by the moments is unique.

Proof. Using Corollary 2.3 and Proposition 2.2, we obtain a differential equation for $S(z)$, namely

$$
rS'(z)=rA(z)+2qrzS'(z)A(z)-qrS(z)A(z)
$$

which, in view of Corollary 2.1, leads to the differential equation for $R(z)$ of the form

$$
R'(z)=\frac{R^2(z)\big((1-q)R(z)+q\big)}{R(z)\big(1-p-2qz\big)+p}
$$

with the initial condition $R(0)=1$. Let us observe now that the function $R(z)$ must be symmetric with respect to p and q (this easily follows from the definition of $R(z)$ if we reverse the order in all ordered non-crossing pair partitions over which the summation is taken). Thus, if we denote $R(z)=R_{p,q}(z)$, then $R_{p,q}(z)=R_{q,p}(z)$. Therefore, we obtain another differential equation for $R(z)$ with p and q interchanged. This allows us then to reduce the above equation to the (algebraic) quadratic equation, namely

$$
AR^2(z)+BR(z)+C=0
$$

where

$$
A \;=\; (1-q)^2-(1-p)^2-2pz+2qz\,,
$$

$$
\begin{aligned}
B &= 2q(1-q) - 2p(1-p)\,, \\
C &= q^2 - p^2\,,
\end{aligned}
$$

which has two solutions,

$$
R(z) = \frac{p+q-1 \pm \sqrt{1-2(p+q)z}}{p+q-2+2z}
$$

but only the one corresponding to the minus sign satisfies the initial condition $R(0) = 1$. If we define the corresponding moment generating function by taking $m_n = r_k$ for $n = 2k$ with odd moments equal to zero, we obtain the function $M(z) = \sum_{n=0}^{\infty} m_n z^n = R(z^2)$ and the corresponding Cauchy transform

$$
G(z) = \frac{1}{z} M\left(\frac{1}{z}\right) = \frac{(p+q-1)z - \sqrt{z^2 - 2(p+q)}}{2-(2-p-q)z^2}\,.
$$

turns out to be the Cauchy transform of the (uniquely determined) Kesten distribution $\mu_{p,q}$ (with the absolutely continuous part given by (1.4)). ■

Remark 2.2. The continued fraction representation of $G(z)$ takes the form

$$
G(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{z - \ddots}}}}\,.
$$

where $t = 1/2(p+q)$. Such Cauchy transforms were obtained in the context of the so-called t-transformation of measures [7]. However, we will show later that our combinatorics is different and, when carried over to the Fock space level, gives a different Brownian motion (although it also has Kesten distributions).

3. Noncommutative Brownian motions

We will now construct new types of noncommutative Brownian motions on the free Fock space which have Kesten distributions and are parametrized by two nonnegative real numbers p, q. They can be viewed as an interpolation between the free Brownian motion obtained for $(p,q) = (1,1)$ and the monotone Brownian motion corresponding to $(p,q) = (0,1)$ (the anti-monotone and boolean Brownian motions are also obtained, for $(p,q) = (1,0)$ and $(p,q) = (0,0)$, respectively). Moreover, the mixed moments of the associated creation and annihilation operators in the vacuum state agree with their counterparts in free probability and monotone probability, a feature absent in other interpolations.

By the free Fock space over a Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$ we understand the direct sum

$$
\mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n} \cong \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+^n) \tag{3.1}
$$

with the canonical inner product.

Let w be the weight function on $\mathbb{R}_+ \times \mathbb{R}_+$ given by

$$w(s,t) = \begin{cases} p & \text{if } \ 0 < s < t \\ q & \text{if } \ 0 < t < s \\ 1 & \text{otherwise} \end{cases},$$

where $p > 0$, $q \geqslant 0$. Now, introduce special vectors in $\mathcal{F}(\mathcal{H})$ denoted by the 'tensor-like' symbol $f_1 \circledast f_2 \circledast \ldots \circledast f_n$, where

$$(f_1 \circledast f_2 \circledast \ldots \circledast f_n)(t_1, t_2, \ldots, t_n) := f_1(t_1)\sqrt{w(t_1,t_2)}f_2(t_2)\ldots\sqrt{w(t_{n-1},t_n)}f_n(t_n).$$

These vectors remind simple tensors, but they have the 'nearest neighbor coupling'. Note that the set of such vectors is dense in $\mathcal{F}(\mathcal{H})$ if $p > 0$ and $q > 0$. If $(p,q) = (0,1)$, it is dense in the monotone Fock space

$$\mathcal{M}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} L^2(\Delta^{(n)})$$

where $\Delta^{(n)} = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n; t_1 \leqslant \ldots \leqslant t_n\}$. In turn, if $(p,q) = (1,0)$, it is dense in the anti-monotone Fock space (similar to the monotone Fock space, but with the reversed order of coordinates). If $(p,q) = (0,0)$, we obtain in turn $\mathbb{C}\Omega \oplus L^2(\mathbb{R}_+)$.

Let us define suitable creation, gauge and annihilation operators which are (p,q)-deformations of their free counterparts.

Definition 3.1. Define the (p,q)-*creation operator* $a(f) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ associated with $f \in \mathcal{H}$ as the bounded linear extension of

$$a(f)\,\Omega = f \tag{3.2}$$

$$a(f)\ (f_1 \circledast f_2 \circledast \ldots \circledast f_n) = f \circledast f_1 \circledast \ldots \circledast f_n \tag{3.3}$$

where $f, f_1, \ldots, f_n \in \mathcal{H}$.

Definition 3.2. For any $f \in L^\infty(\mathbb{R}_+)$ with $|f(0)| < \infty$, by the (p,q)-*gauge operator* on $\mathcal{F}(\mathcal{H})$ associated with f we understand the bounded linear extension of

$$\begin{aligned} M(f)\,\Omega &= f(0)\Omega \\ M(f)\ (f_1 \circledast f_2 \circledast \ldots \circledast f_n) &= (ff_1) \circledast f_2 \circledast \ldots \circledast f_n \end{aligned}$$

for any $f_1, f_2, \ldots, f_n \in \mathcal{H}$.

In particular, by $M(f,g)$ we will denote the gauge operator associated with the function defined by the weighted inner product on $\mathcal{H}$, namely

$$\langle\langle f, g\rangle\rangle(t) := \int_{\mathbb{R}_+} f(s)\overline{g(s)}w(s,t)ds\,.$$

Note that this gauge operator multiplies the vacuum vector by the value of the inner product of f and g, i.e.

$$M(f,g)\,\Omega = \langle f, g\rangle\,\Omega \tag{3.4}$$

thanks to our assumption that $w(s,0) = 1$ for any $s \geqslant 0$. The (p,q)-gauge operator allows us to write a simple formula for the action of the annihilation operators.

Proposition 3.1. *The action of the (p,q)-annihilation operator $a^*(f)$ associated with $f \in \mathcal{H}$, adjoint with respect to the creation operator $a(f)$, is given by the bounded linear extension of*

$$\begin{aligned} a^*(f)\,\Omega &= 0 \\ a^*(f)\,(f_1 \circledast f_2 \circledast \ldots \circledast f_n) &= M(f_1,f)\,(f_2 \circledast f_3 \circledast \ldots \circledast f_n\,,) \end{aligned} \tag{3.5}$$

where $f_1, f_2, \ldots, f_n \in \mathcal{H}$. Thus, in particular, $a^*(f)f_1 = \langle f_1, f\rangle$.

Remark 3.1. Equivalently, we can use a deformed inner product on the free Fock space and define the creation operators to be the free creation operators, whereas the annihilation operators to be their adjoints with respect to the (p,q)-deformed inner product given by

$$\langle F, G\rangle = \delta_{n,m}\sum_{\sigma\in S_n} w(\sigma^{-1})\int_{\Delta_\sigma} F(t_1,\ldots,t_n)\overline{G(t_1,\ldots,t_n)}dt_1\ldots dt_n\,,$$

for any $p,q>0$, where $F\in L^2(\mathbb{R}_+^n)$, $G\in L^2(\mathbb{R}_+^m)$, and $\langle\Omega,\Omega\rangle = 1$, $\langle \Omega, F,\rangle = 0$. This definition can be extended to $p,q \geqslant 0$ except that one has to divide the above vector space by the corresponding kernel of the sesquilinear form.

The gauge operators (which commute among themselves) allow us to write relations between creation, annihilation and gauge operators in a simple form as the proposition given below shows (we omit the elementary proof).

Proposition 3.2. *The following relations hold:*

$$a^*(g)a(f) = M(f,g), \quad M(h)a(f) = a(hf), \quad a^*(f)M(\overline{h}) = a^*(fh) \tag{3.6}$$

where $f,g\in\mathcal{H}$ and $h \in L^\infty(\mathbb{R}_+)$ with $|h(0)|<\infty$.

By the (p,q)-*Gaussian operator* associated with the function $f\in\mathcal{H}$ we will understand the self-adjoint position operator given by the sum $\omega(f) = a^*(f) + a(f)$. In turn, by the associated *Brownian motion* we will understand the process $(\omega_t)_{t\geqslant 0}$, where $\omega_t = \omega(\chi_{[0,t)})$. Below we will find a formula for the mixed moments $\varphi(\omega(f_1)\omega(f_2)\ldots\omega(f_n))$, where φ is the vacuum state on $\mathcal{F}(\mathcal{H})$ and $f_1,f_2,\ldots,f_n\in\Theta$, where

$$\Theta := \{\chi_{[s,t)}; 0\leqslant s<t<\infty\} \tag{3.7}$$

is the set of characteristic functions of intervals.

We shall assume that supports of these functions are pairwise disjoint and are ordered by the partial order $I_1 < I_2$ whenever $t_1<t_2$ for all $t_1\in I_1$, $t_2\in I_2$. We then set $I_1\leqslant I_2$ whenever $I_1<I_2$ or $I_1 = I_2$. The same notation will be used for the corresponding characteristic functions $f=\chi_{I_1},\ g=\chi_{I_2}$, i.e. $f<g$ and $f\leqslant g$. Note that, as in the monotone case, it is not possible to obtain a similar formula for arbitrary functions, or even for characteristic functions with arbitrary supports.

Example 3.1. Let $f := f_1 = f_2 = f_5 = f_6 = \chi_{[1,2)}$ and $g := f_3 = f_4 = \chi_{[0,1)}$. Besides, to simplify notation, we set $a^\epsilon(f_i) = a_i^\epsilon$ for $i=1,\ldots,6$ and $\epsilon = 1,*$. Then

$$\varphi(\omega(f_1)\omega(f_2)\ldots\omega(f_6)) \;=\; \varphi(a_1^*a_2a_3^*a_4a_5^*a_6) + \varphi(a_1^*a_2^*a_3a_4a_5^*a_6)$$

$$+\varphi(a_1^*a_2a_3^*a_4^*a_5a_6)+\varphi(a_1^*a_2^*a_3a_4^*a_5a_6)$$
$$+\varphi(a_1^*a_2^*a_3^*a_4a_5a_6)\,, \tag{3.8}$$

since the other mixed moments certainly vanish. However, the second, third and fourth summands also give zero contribution to (3.8) since in each of them a creation operator associated with g is paired with an annihilation operator associated with f or vice versa and these functions have disjoint supports. Therefore, it is enough to compute the contribution from the first and last summands:

$$\begin{aligned}\varphi(a_1^*a_2^*a_3^*a_4a_5a_6) &= \langle a_1^*a_2^*a_3^*(g\circledast f\circledast f),\Omega\rangle = p\big\langle a_1^*a_2^*(f\circledast f),\Omega\big\rangle\\ &= p\int_1^2\int_1^2 w(t_1,t_2)dt_1dt_2 = \frac{pq+p}{2}\,.\\ \varphi(a_1^*a_2a_3^*a_4a_5^*a_6) &= 1.\end{aligned}$$

Thus

$$\varphi(\omega(f_1)\omega(f_2)\ldots\omega(f_6)) = \frac{pq+p+2}{2}\,.$$

Before we find a formula for all moments, let us introduce some additional notations (most of them are taken from [16]).

Definition 3.3. Let $f_1,f_2,\ldots,f_n\in\Theta$ have pairwise identical or disjoint supports. We will say that $P=(P_1,\ldots,P_m)\in\mathcal{OP}_n$ is *adapted* to $(f_1,f_2,\ldots,f_n)$, which we denote $P\sim(f_1,f_2,\ldots,f_n)$, if and only if it satisfies two conditions:
1. $i,j\in P_k \implies f_i=f_j\,,$
2. $i\in P_k,\ j\in P_l$ and $k<l \implies f_i\leqslant f_j\,.$

In turn, if $\pi=\{\pi_1,\ldots,\pi_m\}\in\mathcal{P}_n$, then we will say that π is *adapted* to $(f_1,f_2,\ldots,f_n)$ if and only if its blocks satisfy only the first condition, which we denote $\pi\sim(f_1,\ldots,f_n)$.

Definition 3.4. If $\pi\sim(f_1,f_2,\ldots,f_n)$, then the *support* of block π_i is $\operatorname{supp}\pi_i=\operatorname{supp}f_j$, for any $j\in\pi_i$. In turn, the *support* of the partition $\pi\in\mathcal{P}_n$ is

$$\operatorname{supp}\pi=\{(t_1,\ldots,t_m);t_k\in\operatorname{supp}\pi_k,\ k=1,\ldots,m\}\,.$$

Finally, for $\pi\in\mathcal{NC}_{2k}^2$ and any $f_1,f_2,\ldots,f_{2k}\in\Theta$ we will also use a simplified notation

$$a_\pi(f_1,f_2,\ldots,f_{2k})=a^{\epsilon_1}(f_1)a^{\epsilon_2}(f_2)\ldots a^{\epsilon_{2k}}(f_{2k})\,, \tag{3.9}$$

where we understand that $\epsilon_i=*$ and $\epsilon_j=1$ whenever $\{i,j\}$ is a block of π and $i<j$.

Lemma 3.1. *If $f_1,f_2,\ldots,f_{2n}\in\Theta$ have pairwise identical or disjoint supports and $\pi\in\mathcal{NC}_{2n}^2$ is not adapted to $(f_1,f_2,\ldots,f_{2n})$, then $\varphi(a_\pi(f_1,f_2,\ldots,f_{2n}))=0$.*

Proof. Since π is not adapted to $(f_1,\ldots,f_{2n})$, there exists a block $\{i,j\}\in\pi$, such that f_i,f_j have disjoint supports. Assuming that $i<j$, we obtain

$$\begin{aligned}&\varphi\left(a_\pi(f_1,f_2,\ldots,f_{2n})\right)\\ =\ &\langle a^{\epsilon_1}(f_1)\ldots a^*(f_i)\ldots a(f_j)\ldots a^{\epsilon_{2n}}(f_{2n})\Omega,\Omega\rangle\\ =\ &\langle a^{\epsilon_1}(f_1)\ldots a^*(f_i)\ldots a(f_j)(g_1\circledast\ldots\circledast g_k),\Omega\rangle\\ =\ &\langle a^{\epsilon_1}(f_1)\ldots a^*(f_i)\ldots a^{\epsilon_{j-1}}(f_{j-1})(f_j\circledast g_1\circledast\ldots\circledast g_k),\Omega\rangle\\ =\ &c\,\langle a^{\epsilon_1}(f_1)\ldots a^*(f_i)(f_j\circledast g_1\circledast\ldots\circledast g_k),\Omega\rangle\end{aligned}$$

$$
\begin{aligned}
&= c\big\langle a^{\epsilon_1}(f_1)\ldots a^{\epsilon_{i-1}}(f_{i-1})M(f_j,f_i)\,g_1\circledast\ldots\circledast g_k,\Omega\big\rangle \\
&= 0\,,
\end{aligned}
$$

where we used the fact that $M(f_j,f_i)=0$ since f_j and f_i have disjoint supports. ■

Let $\pi=\{\pi_1,\ldots,\pi_k\}\sim(f_1,\ldots,f_n)$, where $f_1,f_2,\ldots,f_n\in\Theta$ have pairwise identical or disjoint supports and let $\sigma\in S_k$. Then the pair (π,σ) can be identified with an ordered partition, for which we can define the set $\Delta_{(\pi,\sigma)}=\operatorname{supp}\pi\cap\Delta_\sigma$, i.e.

$$
\Delta_{(\pi,\sigma)}=\big\{(t_1,\ldots,t_n)\in\mathbb{R}^n;t_{\sigma(1)}\leqslant\ldots\leqslant t_{\sigma(n)}\,,\ t_i\in\operatorname{supp}\pi_i\,,\ i=1,\ldots,n\big\}\,. \tag{3.10}
$$

Then the following Proposition holds.

Proposition 3.3. *Suppose $f_1,\ldots,f_{2n}\in\Theta$ have pairwise identical or disjoint supports and let $\pi=\{\pi_1,\ldots,\pi_n\}$ be a pair partition which is adapted to $(f_1,\ldots,f_{2n})$. Then, for any $\sigma\in S_n$ such that $(\pi,\sigma)\nsim(f_1,\ldots,f_{2n})$ it holds that $\Delta_{(\pi,\sigma)}=\varnothing$.*

Proof. We know that $\pi\sim(f_1,\ldots,f_{2n})$ and $(\pi,\sigma)\nsim(f_1,\ldots,f_{2n})$. Therefore, (π,σ) deos not satisfy condition 2 of Definition 3.3, i.e.

$$
\exists_{k,l\in\{1,\ldots,n\}}\ \forall_{i\in\pi_{\sigma(k)},j\in\pi_{\sigma(l)}}\ k<l\ \text{ i }\ f_i>f_j.
$$

Let us suppose that $(t_1,\ldots,t_n)\in\Delta_{(\pi,\sigma)}$. Then it must hold that

$$
t_{\sigma(k)}\leqslant t_{\sigma(l)}\,, \tag{3.11}
$$

since $k<l$. On the other hand, $f_i>f_j$ and $t_{\sigma(k)}\in\operatorname{supp}\pi_{\sigma(k)}=\operatorname{supp}f_i$ as well as $t_{\sigma(l)}\in\operatorname{supp}\pi_{\sigma(l)}=\operatorname{supp}f_j$, thus $t_{\sigma(k)}>t_{\sigma(l)}$, which contradicts (3.11). ■

In the sequel we will have to collect functions with the same supports in a suitable way. Suppose that $f_1,\ldots,f_{2n}\in\Theta$ have pairwise identical or disjoint supports and let $g_1,\ldots,g_r\in\Theta$ be such that

$$
\{f_1,\ldots,f_{2n}\}=\{g_1,\ldots,g_r\}\ \text{ and }\ g_1<\ldots<g_r \tag{3.12}
$$

and the corresponding (ordered) intervals by $I_1<I_2<\ldots<I_r$. Then we can introduce numbers

$$
b_i=\frac{1}{2}\big|\{j\,;\operatorname{supp}f_j=\operatorname{supp}g^{(i)}\}\big|\,,\qquad i=1,\ldots,r\,. \tag{3.13}
$$

Of course, $b_1+b_2+\ldots+b_r=n$. If there exists a pair partition which is adapted to $(f_1,\ldots,f_{2n})$, then numbers b_i are integers and the following easy proposition holds.

Proposition 3.4. *If $f_1,\ldots,f_{2n}\in\Theta$ have pairwise identical or disjoint supports and $(\pi,\sigma)=(\pi_{\sigma(1)},\pi_{\sigma(2)},\ldots,\pi_{\sigma(n)})\in\mathcal{ONC}^2_{2n}$ is adapted to $(f_1,\ldots,f_{2n})$, then*

$$
\lambda(\Delta_{(\pi,\sigma)})=\prod_{i=1}^{r}\frac{(\lambda(I_i))^{b_i}}{b_i!}
$$

where λ denotes the (1-dimensional as well as n-dimensional) Lebesgue measure.

Proof. Since the volume of $\Delta_{(\pi,\sigma)}$ does not depend on σ, we have

$$
\lambda(\Delta_{(\pi,\sigma)})=\int_{\Delta_{(\pi,\mathrm{id})}}dt_1\ldots dt_n=\prod_{i=1}^{r}\frac{(\lambda(I_i))^{b_i}}{b_i!}
$$

which gives the assertion. ∎

Theorem 3.1. *If $f_1, f_2, \ldots, f_n \in \Theta$ have pairwise identical or disjoint supports, then*

$$\varphi(\omega(f_1)\omega(f_2)\ldots\omega(f_n)) = \prod_{i=1}^{r} \frac{(\lambda(I_i))^{b_i}}{b_i!} \sum_{\substack{P\in\mathcal{ONC}_n^2 \\ P\sim(f_1,\ldots,f_n)}} w(P)\,. \tag{3.14}$$

Proof. Of course, if n is odd, the above moments vanish, which gives the assertion since (3.14) is a sum over the empty set. Assume therefore that $n = 2k$. First, observe that Lemma 3.1 gives

$$\begin{aligned}
\varphi(\omega(f_1)\omega(f_2)\ldots\omega(f_{2k})) &= \sum_{\pi\in\mathcal{NC}_{2k}^2} \varphi\left(a_\pi(f_1, f_2, \ldots, f_{2k})\right) \\
&= \sum_{\substack{\pi\in\mathcal{NC}_{2k}^2 \\ \pi\sim(f_1,\ldots,f_{2k})}} \varphi\left(a_\pi(f_1, f_2, \ldots, f_{2k})\right)\,.
\end{aligned}$$

If $\pi = \{\pi_1, \ldots, \pi_k\} \in \mathcal{NC}_{2k}^2$, then to each number $i \in \{1, \ldots, k\}$ we can assign the number $l_i \in \{0, 1, \ldots, k\}$, in such a way that block π_{l_i} is a neighboring outer block of π_i whenever π_i has any outer blocks, and otherwise we set $l_i = 0$ and $t_0 = 0$. Notice that

$$\varphi\left(a_\pi(f_1, f_2, \ldots, f_{2k})\right) = \int_{\operatorname{supp}\pi} w(t_1, t_{l_1})\ldots w(t_k, t_{l_k})dt_1\ldots dt_k\,.$$

Therefore, we have

$$\begin{aligned}
\varphi\left(\omega(f_1)\omega(f_2)\ldots\omega(f_{2k})\right) &= \sum_{\substack{\pi\in\mathcal{NC}_{2k}^2 \\ \pi\sim(f_1,\ldots,f_{2k})}} \int_{\operatorname{supp}\pi} w(t_1, t_{l_1})\ldots w(t_k, t_{l_k})dt_1\ldots dt_k \\
&= \sum_{\substack{(\pi,\rho)\in\mathcal{ONC}_{2k}^2 \\ (\pi,\rho)\sim(f_1,\ldots,f_{2k})}} \int_{\Delta_{(\pi,\rho)}} w(t_1, t_{l_1})\ldots w(t_k, t_{l_k})dt_1\ldots dt_k \\
&= \prod_{i=1}^{r} \frac{(\lambda(I_i))^{b_i}}{b_i!} \sum_{\substack{P\in\mathcal{ONC}_{2k}^2 \\ P\sim(f_1,\ldots,f_{2k})}} w(P)\,,
\end{aligned}$$

using Propositions 3.3 and 3.4. ∎

Remark 3.2. Note that there are two reasons why the mixed moments (3.14) depend on the supports of the f_i's. The first one is the presence of the lenghts of intervals $I_1, \ldots, I_r$ (in fact, in order that the RHS be not zero, each f_i must appear an even number of times and therefore each lenght can be replaced by an inner product). The second one is the fact that the summation runs only over these P which are adapted to $(f_1, f_2, \ldots, f_n)$. For instance, if $f_1 < f_2 < \ldots < f_m$ and $g_1 > g_2 > \ldots > g_m$, we obtain factorizations

$$\begin{aligned}
\varphi(\omega(f_1)\ldots\omega(f_m)\omega(f_m)\ldots\omega(f_1)) &= q^{n-1}\varphi(\omega^2(f_1))\ldots\varphi(\omega^2(f_n)) \\
\varphi(\omega(g_1)\ldots\omega(g_m)\omega(g_m)\ldots\omega(g_1)) &= p^{n-1}\varphi(\omega^2(g_1))\ldots\varphi(\omega^2(g_n)).
\end{aligned}$$

	(p,q)-interpolation	t-deformation	$t \to (p+q)/2$
$a^*aa^*aa^*a$	1	1	1
$a^*a^*aaa^*a$	$(p+q)/2$	t	$(p+q)/2$
$a^*aa^*a^*aa$	$(p+q)/2$	t	$(p+q)/2$
$a^*a^*aa^*aa$	$(p^2+pq+q^2)/3$	t^2	$(p^2+2pq+q^2)/4$
$a^*a^*a^*aaa$	$(p^2+4pq+q^2)/6$	t^2	$(p^2+2pq+q^2)/4$
$\sum$	$((p+q)^2+2p+2q+2)/2$	$2t^2+2t+1$	$((p+q)^2+2p+2q+2)/2$

TABLE 1. Comparison of mixed moments.

In particular, if $q=1$, the first property reflects the so-called 'pyramidal factorization' of the mixed moments [12]. In fact, it is not hard to see that for $q=1$ we get pyramidal factorization for all mixed moments of type $\varphi(c_1 \ldots c_n d_n \ldots d_1)$, where c_i, d_i are arbitrary elements of the unital algebra $\mathcal{A}_i = \langle 1, a(f_i), a^*(f_i)\rangle$, $1 \leqslant i \leqslant n$. If $q \neq 1$, the first formula can be generalized to a 'non-unital q-pyramidal factorization', in which $c_i, d_i \in \langle a(f_i), a^*(f_i)\rangle$. A similar extension of the second formula is also possible. In fact, one can show that for $q=1$ our processes satisfy all three conditions of the so-called 'generalized Brownian motion' given in [6] (pyramidal factorization, stationarity and gaussianity).

Example 3.2. In this example we will evaluate the sixth moment of $\omega(f)$ for $f = \chi_{[0,1)}$ by computing the contributions associated with all partitions $P \in \mathcal{ONC}_6^2$ corresponding to products of creation and annihilation operators. These contributions will be compared with their counterparts obtained for the moments of t-deformed operators studied in [7]. We have

$$\begin{aligned}\varphi(\omega^6(f)) &= \varphi(a^*aa^*aa^*a) + \varphi(a^*a^*aaa^*a) + \varphi(a^*aa^*a^*aa) \\ &\quad + \varphi(a^*a^*aa^*aa) + \varphi(a^*a^*a^*aaa).\end{aligned}$$

These mixed moments of creation and annihilation operators are given in Table 1. Analogous computations can be done for t-deformed creation and annihilation operators. In Table 1 we compare the values of the summands in the above equation for the (p,q)-interpolation and for the t-deformation. The main observation is that, in general, the mixed moments of (p,q)-free creation and annihilation operators are different than the corresponding mixed moments of t-deformed operators. In particular, for $p=0$ (arcsine law), the t-deformed moments do not reproduce the moments in the monotone case.

Moreover, we can compare two combinatorial formulas for the moments. It follows from [7] that the even moments of Kesten laws (1.4) satisfy the equation

$$\mu_{2n} = \sum_{\pi \in \mathcal{NC}_{2n}^2} t^{\mathrm{in}(\pi)} = \sum_{k=0}^{n-1} \mathcal{D}(n,k) t^k \tag{3.15}$$

for $n \geqslant 1$, where $t = (p+q)/2$ and $\mathrm{in}(\pi)$ is the number of inner blocks in π and the $\mathcal{D}(n,k)$ are the so-called *Delaney's numbers*, which give the numbers of pair partitions in $\mathcal{NC}^2_{2n}$ which have exactly k inner blocks. They are given by the explicit formula

$$\mathcal{D}(n,k) = \binom{n+k-1}{k} - \binom{n+k-1}{k-1},$$

where $k \in \{0,1,\ldots,n-1\}$, $n \geqslant 1$, and $\binom{n}{-1} = 0$.

Definition 3.5. For $n \geqslant 1$, the numbers of partitions from the set $\mathcal{ONC}^2_{2n}$ which have exactly k disorders and j orders will be called *generalized Euler's numbers* and will be denoted $\mathcal{E}(n,k,j)$, i.e.

$$\mathcal{E}(n,k,j) = \left|\{P \in \mathcal{ONC}^2_{2n}; e(P) = k \text{ and } e'(P) = j\}\right|$$

where $0 \leqslant j,k \leqslant n-1$.

Using [7] and the results of this Section, we can find a relation between Delaney's numbers and generalized Euler's numbers.

Proposition 3.5. *For* $n \geqslant 1$ *and* $k \in \{0,1,\ldots,n-1\}$, *it holds that*

$$\mathcal{E}(n,k,j) = \frac{n!}{2^{k+j}}\binom{k+j}{k}\mathcal{D}(n,k+j)\,.$$

Proof. Substituting $t = (p+q)/2$ in (3.15) and performing elementary algebraic calculations, we obtain

$$\mu_{2n} = \sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\frac{\mathcal{D}(n,l)}{2^l}\binom{l}{k}(p+q)^k\,,$$

for $n \geqslant 1$. On the other hand, we know that

$$\mu_{2n} = \frac{1}{n!}\sum_{P \in \mathcal{ONC}^2_{2n}} p^{e(P)} = \sum_{k,j=0}^{n-1}\frac{\mathcal{E}(n,k,j)}{n!}(p+q)^k$$

Comparing the coefficients of these two polynomials, we get the desired relation. ■

4. Central limit theorem on discrete free Fock space

In this section we will define discrete free creation and annihilation operators on the free Fock space $\mathcal{F}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ with a fixed orthonormal basis $\{e_i\}_{i=1}^{\infty}$ (for a discussion of the discrete free Fock space, see [23]). Using them, we will formulate an elementary version of the central limit theorem for '(p,q)-independent' random variables. An abstract treatment of the notion of independence involved here goes beyond the scope of this article and will be given in a separate paper.

Using the infinite matrix

$$w_{i,j} = \begin{cases} p & \text{if } i<j \\ q & \text{if } i>j \\ 1 & \text{if } i=j \end{cases},$$

we define a rescaled orthogonal basis in $\mathcal{F}(\mathcal{H})$,

$$e_{i_1} \circledast e_{i_2} \circledast \ldots \circledast e_{i_n} = \sqrt{w_{i_1,i_2}w_{i_2,i_3}\ldots w_{i_{n-1},i_n}}\, e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}$$

where $i_1, i_2, \ldots, i_n \in \mathbb{N}$. It is easy to see that

$$\langle e_{i_1} \circledast e_{i_2} \circledast \ldots \circledast e_{i_n}, e_{j_1} \circledast e_{j_2} \circledast \ldots \circledast e_{j_n} \rangle = \delta_{i_1,j_1}\delta_{i_2,j_2}\ldots\delta_{i_n,j_n} w_{i_1,i_2} w_{i_2,i_3} \ldots w_{i_{n-1},i_n}$$

Using this basis, we define the creation operators $A_i, i \in \mathbb{N}$ by equations

$$\begin{aligned} A_i(\Omega) &= e_i \\ A_i\left(e_{i_1} \circledast e_{i_2} \ldots \circledast e_{i_n}\right) &= e_i \circledast e_{i_1} \circledast \ldots \circledast e_{i_n} \end{aligned}$$

with their adjoints, annihilation operators, acting as follows:

$$\begin{aligned} A_i^*(\Omega) &= 0 \\ A_i^*(e_j) &= \delta_{i,j}\Omega \\ A_i^*\left(e_{i_1} \circledast e_{i_2} \circledast \ldots \circledast e_{i_n}\right) &= w_{i_1,i_2}\delta_{i,i_1}\ \left(e_{i_2} \circledast e_{i_3} \circledast \ldots \circledast e_{i_n}\right) \end{aligned}$$

We will study the asymptotic behavior of normalized sums

$$S_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \omega_i\,, \tag{4.1}$$

where $\omega_i = A_i + A_i^*$ for $i \in \mathbb{N}$. It is easy to see that the position operators ω_i have mean zero and variance one with respect to the vacuum state φ on $\mathcal{F}(\mathcal{H})$. They will play the role of 'independent' random variables in the central limit theorem. In the propositions given below we state their properties which are rather standard in the central limit context.

Proposition 4.1. *If, among the indices $i_1, i_2, \ldots, i_n \in \mathbb{N}$, there exists i_j, $j \in [n]$, such that $i_j \neq i_k$ for and $k \neq j$, then $\varphi(\omega_{i_1}\ldots\omega_{i_n}) = 0$. If, in turn, $(i_1, i_2, \ldots, i_{2n})$ is a sequence of indices associated with a partition $P \in \mathcal{OP}_{2n}^2$, then*

$$\varphi(\omega_{i_1}\ldots\omega_{i_{2n}}) = \begin{cases} w(P) & \text{if } P \in \mathcal{ONC}_{2n}^2 \\ 0 & \text{if } P \notin \mathcal{ONC}_{2n}^2 \end{cases}.$$

Proof. The first assertion is the usual singleton condition, which clearly holds in our case. In second assertion is obvious if $P \notin \mathcal{ONC}_{2n}^2$ (it easily follows from the definition of the A_i and the A_i^*). Suppose that $P \in \mathcal{ONC}_{2n}^2$. Then there exists r such that $i_r = i_{r+1} \neq i_{r+2} \neq \ldots \neq i_n$. Therefore

$$\begin{aligned} \varphi(\omega_{i_1}\ldots\omega_{i_n}) &= \langle \omega_{i_1}\ldots\omega_{i_{r-1}} A_{i_r}^* A_{i_r} \omega_{i_{r+2}}\ldots\omega_{i_n}\Omega, \Omega\rangle \\ &= w_{i_{r+1},i_{r+2}}\langle \omega_{i_1}\ldots\omega_{i_{r-1}}\omega_{i_{r+2}}\ldots\omega_{i_n}\Omega, \Omega\rangle \\ &= w_{i_{r+1},i_{r+2}}\varphi(\omega_{i_1}\ldots\omega_{i_{r-1}}\omega_{i_{r+2}}\ldots\omega_{i_n}), \end{aligned}$$

where $w_{i_{r+1},i_{r+2}}$ is equal to p or q, depending on whether $i_{r+1} < i_{r+2}$ (then block $\{r, r+1\}$ forms a disorder with its neighboring outer block) or $i_{r+1} > i_{r+2}$ (then block $\{r, r+1\}$ forms an order with its neighboring outer block), respectively, and otherwise $w_{i_{r+1},i_{r+2}} = 1$ (then block $\{r, r+1\}$ does not have outer blocks). The assertion follows from induction. ■

Theorem 4.1. *It holds that*

$$\lim_{N\to\infty} \varphi\left(S_N^{2n}\right) = \frac{1}{n!} \sum_{P\in\mathcal{ONC}_{2n}^2} w(P)$$

and the odd moments vanish in the limit.

Proof. The proof is standard since the mixed moments of the ω_i are invariant under order preserving injections, which gives

$$\varphi\left(S_N^{2n}\right) = \frac{1}{N^n} \sum_{i_1,\ldots,i_{2n}} \varphi(\omega_{i_1}\omega_{i_2}\ldots\omega_{i_{2n}}) = \frac{1}{N^n} \sum_{r=1}^{2n} \binom{N}{r} \sum_{P\in\mathcal{OP}_{2n}(r)} \varphi(\omega_P)\,.$$

where $\mathcal{OP}_{2n}(r)$ is the set of ordered partitions of the set $[n]$ which have r blocks and $\varphi(\omega_P)$ denotes $\varphi(\omega_{i_1}\omega_{i_2}\ldots\omega_{i_n})$ for any sequence $(i_1, i_2, \ldots, i_n)$ associated with P. Using Proposition 4.1 and standard arguments, we obtain the assertion. ∎

5. Poisson processes

In this Section we shall introduce processes of Poisson type, denoted $(\gamma_t)_{t\geqslant 0}$ which correspond to the (p, q)-Brownian motion studied in the previous Section. The moments of γ_t in the vacuum state on $\mathcal{F}(\mathbb{R}_+)$ are given by the combinatorial formula

$$\varphi(\gamma_t^n) = \sum_{P\in\mathcal{ONC}_n} \frac{t^b(P)}{b(P)!} w(P)\,,$$

where $t > 0$ (p, q are supressed in the notation). Again, as in the case of position processes, for $(p, q) = (1, 1)$ we obtain the moments of the free Poisson process [22] given by

$$\varphi(\gamma_t^n) = \sum_{\pi\in\mathcal{NC}_n} t^{b(\pi)}$$

where $b(\pi)$ denotes the number of blocks of π. In turn, for $(p, q) = (0, 1)$ we get the moments of the monotone Poisson process

$$\varphi(\gamma_t^n) = \sum_{P\in\mathcal{MON}_n} \frac{t^b(P)}{b(P)!},$$

see [20]. Therefore, the process $(\gamma_t)_{t\geqslant 0}$ plays the role of a natural interpolation between these two processes.

To construct γ_t we shall use the gauge operator (Definition 3.3) associated with the characteristic function $\chi_{[0,t)}$, namely $m_t = M(\chi_{[0,t)})$, which is given by the explicit formula

$$\begin{aligned} m_t\Omega &= 0\,, \\ m_t\left(f_1 \circledast f_2 \circledast \ldots \circledast f_n\right) &= (\chi_{[0,t)}f_1) \circledast f_2 \circledast \ldots \circledast f_n\,. \end{aligned}$$

Such operators were used also in the free case in a similar context [22], where the Poisson process was defined as $l_t + l_t^* + l_t^* l_t + m_t$, where l_t denotes the free creation operator associated with the function $\chi_{[0,t)}$.

An analogous form of the Poisson process will be adopted for the (p,q)-interpolation, namely

$$\gamma_t = a_t + a_t^* + a_t^* a_t + m_t \tag{5.1}$$

and will be called the (p,q)-*Poisson process*. From Proposition 3.3 it follows that $a_t^* a_t = n_t = M(\chi_{[0,t)}, \chi_{[0,t)})$ is another gauge operator given by the explicit formula

$$\begin{aligned} n_t\Omega &= t\Omega\,, \\ n_t\,(f_1 \circledast f_2 \circledast \ldots \circledast f_n) &= (Wf_1) \circledast f_2 \circledast \ldots \circledast f_n \end{aligned}$$

where $W(s) = \int_0^t w(u,s)du = \big((p-q)s + qt\big)$.

Example 5.1. Let us compute low order moments of the (p,q)-Poisson process.

$$\begin{aligned}
\langle \gamma_t^1 \Omega, \Omega \rangle &= t\,, \\
\langle \gamma_t^2 \Omega, \Omega \rangle &= t + t^2\,, \\
\langle \gamma_t^3 \Omega, \Omega \rangle &= t + \frac{p+q+4}{2}t^2 + t^3\,, \\
\langle \gamma_t^4 \Omega, \Omega \rangle &= t + \frac{3p+3q+6}{2}t^2 + \frac{p^2+pq+q^2+3p+3q}{3}t^3 + t^4\,, \\
\langle \gamma_t^5 \Omega, \Omega \rangle &= t + (3p+3q+4)t^2 + \frac{11p^2+11pq+11q^2+24p+24q+36}{6}t^3 \\
&\quad + \frac{3p^3+3p^2q+3pq^2+3q^2+8p^2+8q^2+18p+18q+48}{12}t^4 + t^5
\end{aligned}$$

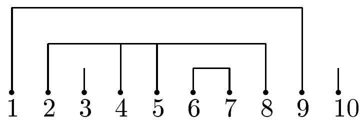
For instance, we get

$$\begin{aligned}
\langle \gamma_t^4 \Omega, \Omega \rangle &= t\,\langle \gamma_t^3 \Omega, \Omega \rangle + \langle \gamma_t^3 \big(\chi(t_1)\big), \Omega \rangle \\
&= t\,\langle \gamma_t^3 \Omega, \Omega \rangle + \langle \gamma_t^2 \big(\chi(t_2)\chi(t_1) + (W(t_1)+1)\chi(t_1) + t\Omega\big), \Omega \rangle \\
&= t\,\langle \gamma_t^3 \Omega, \Omega \rangle + t\,\langle \gamma_t^2 \Omega, \Omega \rangle \\
&\quad + \langle \gamma_t \big((W^2(t_1) + 3W(t_1) + 1)\chi(t_1) + (t + \frac{p+1}{2}t^2)\Omega\big), \Omega \rangle \\
&= t^2 + \frac{p+q+4}{2}t^3 + t^4 + t^2 + t^3 \\
&\quad + \int_0^t (W^2(t_1) + 3W(t_1) + 1)dt_1 + t^2 + \frac{p+q}{2}t^3 \\
&= t + \frac{3p+3q+6}{2}t^2 + \frac{p^2+pq+q^2+3p+3q+9}{3}t^3 + t^4\,.
\end{aligned}$$

where, for simplicity, we denote $\chi = \chi_{[0,t)}$.

Before we compute all moments of γ_t, we introduce some notations. Let $\pi = \{\pi_1, \ldots, \pi_b\} \in \mathcal{NC}_n$ be a partition of $[n]$. Let us divide the set $[n]$ into the following disjoint subsets:

$$\begin{aligned}
\mathfrak{a}_\pi &= \big\{ i \in [n]; \exists_{r \in \{1,\ldots,b\}}\ |\pi_r| > 1,\ i = \max \pi_r \big\} \\
\mathfrak{a}_\pi^* &= \big\{ i \in [n]; \exists_{r \in \{1,\ldots,b\}}\ |\pi_r| > 1,\ i = \min \pi_r \big\} \\
\mathfrak{m}_\pi &= [n] \backslash (\mathfrak{n}_\pi \cup \mathfrak{a}_\pi \cup \mathfrak{a}_\pi^*)
\end{aligned}$$



$$\longrightarrow \qquad c_\pi = a_t^* a_t^* n_t m_t n_t a_t^* a_t a_t a_t n_t$$

FIGURE 4. Partition π and the corresponding operator c_π.

$$\mathfrak{n}_\pi \quad = \quad \left\{ i \in [n]; \exists_{r \in \{1,\ldots,b\}} \ \pi_r = \{i\} \right\}.$$

In other words, the sets $\mathfrak{a}_\pi^*$ and $\mathfrak{a}_\pi$ consist of left and right legs of the blocks of π, respectively, $\mathfrak{m}_\pi$ corresponds to the 'middle' legs of the blocks $\pi_1, \ldots, \pi_b$ and the set $\mathfrak{n}_\pi$ corresponds to singletons in π.

Using these sets, we can assign to each $\pi = \{\pi_1, \ldots, \pi_b\} \in \mathcal{NC}_n$ an operator $c_\pi = c_1 \ldots c_n$, where

$$c_i = \begin{cases} a_t & \text{if } \ i \in \mathfrak{a}_\pi \\ a_t^* & \text{if } \ i \in \mathfrak{a}_\pi^* \\ m_t & \text{if } \ i \in \mathfrak{m}_\pi \\ n_t & \text{if } \ i \in \mathfrak{n}_\pi \end{cases}.$$

Of course, the mapping $\pi \to c_\pi$ is one-to-one, but it is not onto. However, the products of operators a_t, a_t^*, n_t, m_t, which do not correspond to any non-crossing partition π will turn out irrelevant. In Fig.4 we show a non-crossing partition $\pi \in \mathcal{ONC}_{10}(5)$ with the corresponding operator c_π. In turn, examples of products which do not correspond to any non-crossing partitions are given by $m_t m_t m_t$, $a_t^* a_t a_t$, $a_t a_t^* a_t^* a_t$.

Lemma 5.1. *Let $c_1, \ldots, c_n \in \{a_t, a_t^*, m_t, n_t\}$, be a sequence of operators, for which there exists no partition $\pi \in \mathcal{NC}_n$ such that $c_\pi = c_1 \ldots c_n$. Then $\langle c_1 \ldots c_n \Omega, \Omega \rangle = 0$.*

Proof. Observe that if $c_\pi \neq c_1 \ldots c_n$ for all $\pi \in \mathcal{NC}_n$, then one of the following three cases must hold:
1. The number of creation operators a_t in the sequence $c_1, \ldots, c_n$ must be different from the number of annihilation operators a_t^*. Then the assertion is certainly true.
2. There exists $i \in [n]$ such that among $c_i, c_{i+1}, \ldots, c_n$ there are more annihilation oparators a_t^* than creation operators a_t. In that case, the assertion certainly holds.
3. There exists $i \in [n]$ such that $c_i = m_t$ and in the sequence $c_{i+1}, \ldots, c_n$ there are as many annihilation operators a_t^* as creation operators a_t. Then we have

$$\langle c_1 \ldots c_n \Omega, \Omega \rangle = \langle c_1 \ldots c_{i-1} m_t C \Omega, \Omega \rangle = 0\,,$$

for a certain constant C. ■

Theorem 5.1. *The moments of the Poisson process $(\gamma_t)_{t \geqslant 0}$ in the vacuum state are given by*

$$\varphi(\gamma_t^n) = \sum_{P \in \mathcal{ONC}_n} \frac{t^b(P)}{b(P)!} w(P)$$

for any $n \in \mathbb{N}$, and $\varphi(\gamma_t^0) = 1$.

Proof. First, notice that from Lemma 5.1 it folows that

$$\varphi(\gamma_t^n) = \sum_{\pi\in\mathcal{NC}_n} \langle c_\pi\Omega, \Omega\rangle \ ,$$

and therefore it suffices to show that if $\pi \in \mathcal{NC}_n$ has $b(\pi) = b$ blocks, then it holds that

$$\langle c_\pi\Omega, \Omega\rangle = \frac{t^b}{b!}\sum_{\sigma\in S_b} w(\pi,\sigma)$$

where $w(\pi,\sigma) = w(P)$ for $P = (\pi,\sigma)$. For that purpose, let us construct a pair-partition $\pi'' \in \mathcal{NC}^2_{2b}$ such that $\langle c_\pi\Omega, \Omega\rangle = \langle a_{\pi''}\Omega, \Omega\rangle$ and $e(\pi,\sigma) = e(\pi'',\sigma)$, $e'(\pi,\sigma) = e'(\pi'',\sigma)$ for any permutation $\sigma \in S_b$, where $a_{\pi''}$ is the abbreviated notation for $a_{\pi''}(\chi_{[0,t)},\ldots,\chi_{[0,t)})$.

Let us notice that in the product c_π we can omit all occurences of m_t since they correspond to the 'middle' legs of blocks of π, and therefore their action will not change the value of $\langle c_\pi\Omega, \Omega\rangle$. In other words,

$$\langle c_\pi\Omega, \Omega\rangle = \langle\Big(\overrightarrow{\prod_{i\in\mathfrak{a}^*_\pi\cup\mathfrak{a}_\pi\cup\mathfrak{n}^*_\pi}} c_i\Big)\Omega, \Omega\rangle = \langle c_{\pi'}\Omega, \Omega\rangle \ ,$$

where π' is a certain non-crossing partition with b blocks, each consisting of one or two elements. Substituting the operator $a_t^*a_t$ for each m_t in the product $c_{\pi'}$ corresponds to replacing singletons by two-element blocks $\{i, i+1\}$. Therefore, $c_{\pi'} = a_{\pi''}$ for a certain pair-partition $\pi'' \in \mathcal{NC}^2_{2b}$. Moreover, from the construction of π'' it follows that $e(\pi,\sigma) = e(\pi'',\sigma)$ and $e'(\pi,\sigma) = e'(\pi'',\sigma)$ for any permutation $\sigma \in S_b$. Now, from the proof of Theorem 3.1 we have

$$\langle c_\pi\Omega, \Omega\rangle = \langle a_{\pi''}\Omega, \Omega\rangle = \frac{t^b}{b!}\sum_{\sigma\in S_b} w(\pi'',\sigma) = \frac{t^b}{b!}\sum_{\sigma\in S_b} w(\pi,\sigma)\,,$$

which completes the proof. ■

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